

# CROFTON FORMULAE FOR TENSOR-VALUED CURVATURE MEASURES

DANIEL HUG AND JAN A. WEIS

**ABSTRACT.** The tensorial curvature measures are tensor-valued generalizations of the curvature measures of convex bodies. We prove a set of Crofton formulae for such tensorial curvature measures. These formulae express the integral mean of the tensorial curvature measures of the intersection of a given convex body with a uniform affine  $k$ -flat in terms of linear combinations of tensorial curvature measures of the given convex body. Here we first focus on the case where the tensorial curvature measures of the intersection of the given body with an affine flat is defined with respect to the affine flat as its ambient space. From these formulae we then deduce some new and also recover known special cases. In particular, we substantially simplify some of the constants that were obtained in previous work on Minkowski tensors. In a second step, we explain how the results can be extended to the case where the tensorial curvature measure of the intersection of the given body with an affine flat is determined with respect to the ambient Euclidean space.

## 1. INTRODUCTION

The *classical Crofton formula* is a major result in integral geometry. Its name originates from works of the Irish mathematician Morgan W. Crofton [4] on integral geometry in  $\mathbb{R}^2$  in the late 19th century. For a convex body  $K$  (a non-empty, convex and compact set) in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , the classical Crofton formula (see [19, (4.59)]) states that

$$(1) \quad \int_{A(n,k)} V_j(K \cap E) \mu_k(dE) = \alpha_{njk} V_{n-k+j}(K),$$

for  $k \in \{0, \dots, n\}$  and  $j \in \{0, \dots, k\}$ , where  $A(n, k)$  is the affine Grassmannian of  $k$ -flats in  $\mathbb{R}^n$ ,  $\mu_k$  denotes the motion invariant Haar measure on  $A(n, k)$ , normalized as in [20, p. 588], and  $\alpha_{njk} > 0$  is an explicitly known constant.

Let  $\mathcal{K}^n$  denote the set of convex bodies in  $\mathbb{R}^n$ . The functionals  $V_i : \mathcal{K}^n \rightarrow \mathbb{R}$ , for  $i \in \{0, \dots, n\}$ , appearing in (1), are the *intrinsic volumes*, which occur as the coefficients of the monomials in the *Steiner formula*

$$(2) \quad V_n(K + \epsilon B^n) = \sum_{j=0}^n \kappa_{n-j} V_j(K) \epsilon^{n-j},$$

for a convex body  $K \in \mathcal{K}^n$  and  $\epsilon \geq 0$  (cf. [14, (1.16)]); here, as usual,  $+$  denotes the Minkowski addition in  $\mathbb{R}^n$  and  $\kappa_n$  is the volume of the Euclidean unit ball  $B^n$  in  $\mathbb{R}^n$ . Properties of the  $V_i$  such

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2010 *Mathematics Subject Classification.* Primary: 52A20, 53C65; secondary: 52A22, 52A38, 28A75.

*Key words and phrases.* Crofton formula, tensor valuation, curvature measure, Minkowski tensor, integral geometry.

The authors were supported in part by DFG grants FOR 1548 and HU 1874/4-2.

as continuity, isometry invariance and additivity are derived from corresponding properties of the volume. A key result for the intrinsic volumes is *Hadwiger's characterization theorem* (see [7, 2. Satz] and [14, Theorem 1.23]), which states that  $V_0, \dots, V_n$  form a basis of the vector space of continuous and isometry invariant real-valued valuations on  $\mathcal{K}^n$ .

A natural way to extend the Crofton formula is to apply the integration over the affine Grassmannian  $A(n, k)$  to functionals which generalize the intrinsic volumes. One of these generalizations concerns the class of continuous and isometry covariant  $\mathbb{T}^p$ -valued valuations on  $\mathcal{K}^n$ , where  $\mathbb{T}^p$  denotes the vector space of symmetric tensors of rank  $p \in \mathbb{N}_0$  over  $\mathbb{R}^n$ .

The  $\mathbb{T}^0$ -valued valuations are simply the well-known and extensively studied intrinsic volumes. For the  $\mathbb{T}^1$ -valued (i.e. vector-valued) valuations, Hadwiger and Schneider [8, Hauptsatz] proved in 1971 a characterization theorem similar to the aforementioned real-valued case due to Hadwiger. In addition, they also established integral geometric formulae, including a Crofton formula [8, (5.4)]. In 1997, McMullen [16] initiated a systematic investigation of this class of  $\mathbb{T}^p$ -valued valuations for general  $p \in \mathbb{N}_0$ . Only two years later Alesker generalized Hadwiger's characterization theorem (see [2, Theorem 2.2] and [14, Theorem 2.5]) by showing that the vector space of continuous and isometry covariant  $\mathbb{T}^p$ -valued valuations on  $\mathcal{K}^n$  is spanned by the tensor-valued versions of the intrinsic volumes, the *Minkowski tensors*  $\Phi_j^{r,s}$ , where  $j, r, s \in \mathbb{N}_0$  and  $j < n$ , multiplied with suitable powers of the metric tensor in  $\mathbb{R}^n$ . In 2008, Hug, Schneider and Schuster proved a set of Crofton formulae for these Minkowski tensors (see [12, Theorem 2.1–2.6]).

Localizations of the intrinsic volumes yield other types of generalizations. The *support measures* are weakly continuous, locally defined and motion equivariant valuations on convex bodies with values in the space of finite measures on Borel subsets of  $\mathbb{R}^n \times \mathbb{S}^{n-1}$ , where  $\mathbb{S}^{n-1}$  denotes the Euclidean unit sphere in  $\mathbb{R}^n$ . These are determined by a local version of (2). Therefore, they are a crucial example of localizations of the intrinsic volumes. Furthermore, their marginal measures on Borel subsets of  $\mathbb{R}^n$  are called *curvature measures* and the ones on Borel subsets of  $\mathbb{S}^{n-1}$  are called *surface area measures*. In 1959, Federer [5] proved Crofton formulae for curvature measures, even in the more general setting of sets with positive reach. For further details and references, see also [14, Section 1.3] and [14, Section 1.5]. Certain Crofton formulae for support measures were proved by Glasauer in 1997 [6, Theorem 3.2].

The combination of Minkowski tensors and localizations leads to another generalization of the intrinsic volumes. This topic has been explored by Schneider [18] and Hug and Schneider [9, 10] in recent years. They introduced particular tensorial support measures, the *generalized local Minkowski tensors*, and proved that they essentially span the vector space of isometry covariant and locally defined valuations on the space of convex polytopes  $\mathcal{P}^n$  with values in the  $\mathbb{T}^p$ -valued measures on  $\mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$  (see [14, Section 2.4]). Under the additional assumption of weak continuity they extended this result to valuations on  $\mathcal{K}^n$ ; a summary of the required arguments is given in [14, Section 2.5].

The aim of the present article is to prove a set of Crofton formulae for similar functionals, which are localized in  $\mathbb{R}^n$ , the *tensorial curvature measures* or *tensor-valued curvature measures*. Here we first focus on the case where the tensorial curvature measures of the intersection of the given body with an affine flat are defined with respect to the affine flat as the ambient space (intrinsic viewpoint). In a second step, we demonstrate how the arguments can be extended to the case where the curvature

measures are considered in  $\mathbb{R}^n$  (extrinsic viewpoint). The current approach combines main ideas of the previous works [12] and [9] and also links it to [3]. A major advantage of the localization is that it naturally leads to a suitable choice of local tensor-valued measures for which the constants in the Crofton formulae are reasonably simple. From the general local results, we finally deduce various special consequences for the total measures, which are the Minkowski tensors that have been studied in [12]. For the latter, we restrict ourselves to the translation invariant case, which simplifies the involved constants, but the general case can be treated similarly. In the case of the results for the extrinsic tensorial Crofton formulae, the connection to the approach in [3] via the methods of algebraic integral geometry is used and deepened, but this interplay will have to be explored further in future work.

The structure of this contribution is as follows. In Section 2, we fix our notation and collect various auxiliary results which will be needed. Section 3 contains the main results. We first state our findings for intrinsic tensorial curvature measures, then discuss some special cases and finally explain the extension to extrinsic tensorial curvature measures. The proofs of the results for the intrinsic case are given in Section 4. Section 5 contains the arguments in the extrinsic setting. Some auxiliary results on sums of Gamma functions are provided in the final section.

## 2. SOME BASIC TOOLS

In the following, we work in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , equipped with its usual topology generated by the standard scalar product  $\cdot$  and the corresponding Euclidean norm  $\|\cdot\|$ . Recall that the unit ball centered at the origin is denoted by  $B^n$ , its boundary (the unit sphere) is denoted by  $\mathbb{S}^{n-1}$ . For a topological space  $X$ , we denote the Borel  $\sigma$ -algebra on  $X$  by  $\mathcal{B}(X)$ .

By  $G(n, k)$ , for  $k \in \{0, \dots, n\}$ , we denote the Grassmannian of  $k$ -dimensional linear subspaces in  $\mathbb{R}^n$ , and we write  $\nu_k$  for the (rotation invariant) Haar probability measure on  $G(n, k)$ . The directional space of an affine  $k$ -flat  $E \in A(n, k)$  is denoted by  $L(E) \in G(n, k)$ , its orthogonal complement by  $E^\perp \in G(n, n-k)$ , and the translate of  $E$  by a vector  $t \in \mathbb{R}^n$  is denoted by  $E_t := E + t$ . For  $k \in \{0, \dots, n\}$ ,  $l \in \{0, \dots, k\}$  and  $F \in G(n, k)$ , we define  $G(F, l) := \{L \in G(n, l) : L \subset F\}$ . On  $G(F, l)$  there exists a unique Haar probability measure  $\nu_l^F$  invariant under rotations of  $\mathbb{R}^n$  mapping  $F$  into itself and leaving  $F^\perp$  pointwise fixed. The orthogonal projection of a vector  $x \in \mathbb{R}^n$  to a linear subspace  $L$  of  $\mathbb{R}^n$  is denoted by  $p_L(x)$  and its direction by  $\pi_L(x) \in \mathbb{S}^{n-1}$ , if  $x \notin L^\perp$ . For two linear subspaces  $L, L'$  of  $\mathbb{R}^n$ , the generalized sine function  $[L, L']$  is defined as follows. One extends an orthonormal basis of  $L \cap L'$  to an orthonormal basis of  $L$  and to one of  $L'$ . Then  $[L, L']$  is the volume of the parallelepiped spanned by all these vectors.

The vector space of symmetric tensors of rank  $p \in \mathbb{N}_0$  over  $\mathbb{R}^n$  is denoted by  $\mathbb{T}^p$ . The symmetric tensor product of two vectors  $x, y \in \mathbb{R}^n$  is denoted by  $xy$  and the  $p$ -fold tensor product of a vector  $x \in \mathbb{R}^n$  by  $x^p$ . Identifying  $\mathbb{R}^n$  with its dual space via its scalar product, we interpret a symmetric tensor  $a \in \mathbb{T}^p$  as a symmetric  $p$ -linear map from  $(\mathbb{R}^n)^p$  to  $\mathbb{R}$ . One special tensor is the *metric tensor*  $Q \in \mathbb{T}^2$ , defined by  $Q(x, y) := x \cdot y$  for  $x, y \in \mathbb{R}^n$ . For an affine  $k$ -flat  $E \in A(n, k)$ ,  $k \in \{0, \dots, n\}$ , the metric tensor  $Q(E)$  in  $E$  is defined by  $Q(E)(x, y) := p_{L(E)}(x) \cdot p_{L(E)}(y)$  for  $x, y \in \mathbb{R}^n$ .

Defining the tensorial curvature measures requires some preparation (see also [14, Section 1.3]). For a convex body  $K \in \mathcal{K}^n$ , we call the pair  $(x, u) \in \mathbb{R}^{2n}$  a *support element* whenever  $x$  is a

boundary point of  $K$  and  $u$  is an outer unit normal vector of  $K$  at  $x$ . The set of all these support elements of  $K$  is denoted by  $\text{Nor } K \subset \Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$  and called the *normal bundle* of  $K$ . For  $x \in \mathbb{R}^n$ , we denote the metric projection of  $x$  onto  $K$  by  $p(K, x)$ , and define  $u(K, x) := (x - p(K, x)) / \|x - p(K, x)\|$  for  $x \in \mathbb{R}^n \setminus K$ , the unit vector pointing from  $p(K, x)$  to  $x$ . For  $\epsilon > 0$  and a Borel set  $\eta \subset \Sigma^n$ ,

$$M_\epsilon(K, \eta) := \{x \in (K + \epsilon B^n) \setminus K : (p(K, x), u(K, x)) \in \eta\}$$

is a local parallel set of  $K$  which satisfies a *local Steiner formula*

$$(3) \quad V_n(M_\epsilon(K, \eta)) = \sum_{j=0}^{n-1} \kappa_{n-j} \Lambda_j(K, \eta) \epsilon^{n-j}, \quad \epsilon \geq 0.$$

This relation determines the *support measures*  $\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$  of  $K$ , which are finite Borel measures on  $\mathcal{B}(\Sigma^n)$ . Obviously, a comparison of (2) and (3) yields  $V_j(K) = \Lambda_j(K, \Sigma^n)$ .

Now, for a convex body  $K \in \mathcal{K}^n$ , a Borel set  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $j, r, s \in \mathbb{N}_0$ , the *tensorial curvature measures* are given by

$$\phi_j^{r,s,0}(K, \beta) := \omega_{n-j} \int_{\beta \times \mathbb{S}^{n-1}} x^r u^s \Lambda_j(K, d(x, u)),$$

for  $j \in \{0, \dots, n-1\}$ , where  $\omega_n$  denotes the  $n-1$ -dimensional volume of  $\mathbb{S}^{n-1}$ , and by

$$\phi_n^{r,0,0}(K, \beta) := \int_{K \cap \beta} x^r \mathcal{H}^n(dx).$$

If  $K \subset E \in A(n, k)$  with  $j < k \leq n$ , we denote the  $j$ -th support measure of  $K$  defined with respect to  $E$  as the ambient space by  $\Lambda_j^{(E)}(K, \cdot)$ , which is a Borel measure on  $\mathcal{B}(\mathbb{R}^n \times (L(E) \cap \mathbb{S}^{n-1}))$ , concentrated on  $\Sigma^{(E)} := E \times (L(E) \cap \mathbb{S}^{n-1})$  with  $L(E) \in G(n, k)$  being the linear subspace parallel to  $E$ . Then, we define the intrinsic tensorial curvature measures

$$\phi_{j,E}^{r,s,0}(K, \beta) := \omega_{k-j} \int_{\beta \times (L(E) \cap \mathbb{S}^{n-1})} x^r u^s \Lambda_j^{(E)}(K, d(x, u))$$

and

$$\phi_{k,E}^{r,0,0}(K, \beta) := \int_{K \cap \beta} x^r \mathcal{H}^k(dx).$$

For the sake of convenience, we extend the definition by  $\phi_j^{r,s,0} := 0$  (resp.  $\phi_{j,E}^{r,s,0} := 0$ ) for  $j \notin \{0, \dots, n\}$  (resp.  $j \notin \{0, \dots, k\}$ ) or  $r \notin \mathbb{N}_0$  or  $s \notin \mathbb{N}_0$  or  $j = n$  (resp.  $j = k$ ) and  $s \neq 0$ . We adopt the same convention for the Minkowski tensors and the generalized tensorial curvature measures introduced below.

The tensorial curvature measures are natural local versions of the *Minkowski tensors*. For a convex body  $K \in \mathcal{K}^n$  and  $j, r, s \in \mathbb{N}_0$ , the latter are just the total measures  $\Phi_j^{r,s}(K) := \phi_j^{r,s,0}(K, \mathbb{R}^n)$  and, if  $K \subset E \in A(n, k)$ , an intrinsic version is given by  $\Phi_{j,E}^{r,s}(K) := \phi_{j,E}^{r,s,0}(K, \mathbb{R}^n)$ . These definitions of the Minkowski tensors differ slightly from the ones commonly used in the literature, as we slightly change the usual normalization (compare with the normalization used in [14, Definition 2.1]). The

purpose of this change is to simplify the presentation of the main results of this article (and of future work).

For a polytope  $P \in \mathcal{P}^n$  and  $j \in \{0, \dots, n\}$ , we denote the set of  $j$ -dimensional faces of  $P$  by  $\mathcal{F}_j(P)$  and the normal cone of  $P$  at a face  $F \in \mathcal{F}_j(P)$  by  $N(P, F)$ . For a polytope  $P \in \mathcal{P}^n$  and a Borel set  $\eta \subset \Sigma$ , the  $j$ -th support measure is explicitly given by

$$\Lambda_j(P, \eta) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{N(P, F) \cap \mathbb{S}^{n-1}} \mathbf{1}_\eta(x, u) \mathcal{H}^{n-j-1}(du) \mathcal{H}^j(dx)$$

for  $j \in \{0, \dots, n-1\}$ . For  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , this yields

$$\phi_j^{r,s,0}(P, \beta) = \sum_{F \in \mathcal{F}_j(P)} \int_{F \cap \beta} x^r \mathcal{H}^j(dx) \int_{N(P, F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du)$$

and, if  $P \subset E \in A(n, k)$  and  $j < k \leq n$ ,

$$\phi_{j,E}^{r,s,0}(P, \beta) = \sum_{F \in \mathcal{F}_j(P)} \int_{F \cap \beta} x^r \mathcal{H}^j(dx) \int_{N_E(P, F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{k-j-1}(du),$$

where  $N_E(P, F) = N(P, F) \cap L(E)$  is the normal cone of  $P$  at the face  $F$ , taken with respect to the subspace  $L(E)$ . Of course, analogous representations are obtained for the (global) intrinsic and extrinsic Minkowski tensors.

The Crofton formulae, which are stated in the next section, will naturally also involve the *generalized tensorial curvature measures* (see [14, (2.28)])

$$\phi_j^{r,s,1}(P, \beta) := \sum_{F \in \mathcal{F}_j(P)} Q(F) \int_{F \cap \beta} x^r \mathcal{H}^j(dx) \int_{N(P, F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du),$$

for  $j \in \{1, \dots, n-1\}$ , and, if  $P \subset E \in A(n, k)$  and  $0 < j < k \leq n$ ,

$$\phi_{j,E}^{r,s,1}(P, \beta) := \sum_{F \in \mathcal{F}_j(P)} Q(F) \int_{F \cap \beta} x^r \mathcal{H}^j(dx) \int_{N_E(P, F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{k-j-1}(du).$$

Due to Hug and Schneider [9] there exists a weakly continuous extension of the generalized tensorial curvature measures to  $\mathcal{K}^n$ . In fact, they proved such an extension for the generalized local Minkowski tensors, which are measures on  $\mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ . Globalizing this in the  $\mathbb{S}^{n-1}$ -coordinate yields the result for the tensorial curvature measures.

Apart from the easily verified relation

$$(4) \quad \phi_{n-1}^{r,s,1} = Q \phi_{n-1}^{r,s,0} - \phi_{n-1}^{r,s+2,0},$$

the tensorial curvature measures and the generalized tensorial curvature measures are linearly independent. In contrast, McMullen [16] discovered basic linear relations for the (global) Minkowski tensors (see also [14, Theorem 2.6]), and it was shown in [13] that these are essentially all linear dependences between the Minkowski tensors (see also [14, Theorem 2.7]). Furthermore, McMullen [16, p. 269] found relations for the global counterparts of the generalized tensorial curvature measures. In fact, the globalized form of (4) is a very special example of one of these relations. For the

translation invariant Minkowski tensors  $\Phi_j^{0,s}$ , these relations take a very simple form, nevertheless for our purpose they are essential in the proof of Theorem 4. To have a short notation for these translation invariant Minkowski tensors, we omit the first superscript and put

$$\Phi_j^s := \Phi_j^{0,s}, \quad \Phi_{j,E}^s := \Phi_{j,E}^{0,s}.$$

Then we can state the following very special case of McMullen's relations.

**Lemma 1** (McMullen). *Let  $P \in \mathcal{P}^n$  and  $j, s \in \mathbb{N}_0$  with  $j \leq n - 1$ . Then*

$$\frac{n-j+s}{s+1} \Phi_j^{s+2}(P) = \sum_{F \in \mathcal{F}_j(P)} Q(F^\perp) \mathcal{H}^j(F) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du).$$

Note that this lemma is essentially a global result which is derived by applying a version of the divergence theorem.

### 3. CROFTON FORMULAE

In this article, for  $0 \leq j \leq k < n$  and  $i, s \in \mathbb{N}_0$ , we are first concerned with the Crofton integrals

$$(5) \quad \int_{A(n,k)} Q(E)^i \phi_{j,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE),$$

which involve the intrinsic tensorial curvature measures, and the Crofton integrals

$$(6) \quad \int_{A(n,k)} Q(E)^i \Phi_{j,E}^s(K \cap E) \mu_k(dE)$$

for the global versions of the translation invariant intrinsic tensorial curvature measures, the translation invariant intrinsic Minkowski tensors obtained by setting  $r = 0$ . In the global case, we restrict our investigations mainly to these translation invariant intrinsic Minkowski tensors, general Crofton formulae have already been established in [12].

Using the simplifications of the formulae obtained in the present work, the extrinsic formulae in [12], that is, Crofton formulae for the integrals

$$(7) \quad \int_{A(n,k)} \phi_j^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE)$$

can be simplified accordingly. We explain this in detail in the case where  $j = k - 1$ . The connection to [3] turns out to be crucial for simplifying the constants if  $s$  is odd. However, for even  $s$  the current approach works completely independently.

**3.1. Crofton Formulae for Intrinsic Tensorial Curvature Measures.** In this section we state the formulae for the integrals given in (5) and (6). We start with the local versions, where we distinguish the cases  $j = k$  and  $j < k$ .

**Theorem 1.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $i, k, r, s \in \mathbb{N}_0$  with  $k < n$ . Then*

$$\int_{A(n,k)} Q(E)^i \phi_{k,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2} + i)}{\Gamma(\frac{n}{2} + i)\Gamma(\frac{k}{2})} Q^i \phi_n^{r,0,0}(K, \beta)$$

if  $s = 0$ ; for  $s \neq 0$  the integral is zero.

If  $s = 0$  in Theorem 1, then we interpret the coefficient of the tensor on the right-hand side as 0, if  $k = 0$  and  $i \neq 0$ , and as 1, if  $k = i = 0$ . A global version of Theorem 1 is obtained by simply setting  $\beta = \mathbb{R}^n$ .

Next we turn to case  $j < k$ .

**Theorem 2.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $i, j, k, r, s \in \mathbb{N}_0$  with  $j < k < n$  and  $k > 1$ . Then*

$$\begin{aligned} & \int_{A(n,k)} Q(E)^i \phi_{j,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} Q^z(\lambda_{n,k,j,s,i,z}^{(0)} \phi_{n-k+j}^{r,s+2i-2z,0}(K, \beta) + \lambda_{n,k,j,s,i,z}^{(1)} \phi_{n-k+j}^{r,s+2i-2z-2,1}(K, \beta)), \end{aligned}$$

where for  $\varepsilon \in \{0, 1\}$  we set

$$\begin{aligned} \gamma_{n,k,j} &:= \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}, \\ \lambda_{n,k,j,s,i,z}^{(\varepsilon)} &:= \sum_{p=0}^i \sum_{q=(z-p+\varepsilon)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q-\varepsilon}{z} \Gamma(q + \frac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)} \frac{\Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + p + q)} \vartheta_{n,k,j,p,q}^{(\varepsilon)}, \\ \vartheta_{n,k,j,p,q}^{(0)} &:= (n-k+j) \left( \frac{k-1}{2} + p \right), \quad \vartheta_{n,k,j,p,q}^{(1)} := p(n-k) - q(k-1). \end{aligned}$$

If  $j = k - 1$ , then the tensorial curvature measures and the generalized tensorial curvature measures are linearly dependent. In this case, the right-hand side can be expressed as a linear combination of the tensor-valued curvature measures  $Q^z \phi_{n-1}^{r,s+2i-2z,0}(K, \cdot)$ , whereas the measures  $Q^z \phi_{n-1}^{r,s+2i-2z,1}(K, \cdot)$  are not needed. An explicit description of this case is given in Corollary 5 for  $i = 0$  and in (15) for  $i \in \mathbb{N}_0$ .

If the additional metric tensor is omitted as a weight function, that is in the case  $i = 0 = p$ , then the coefficients  $\lambda_{n,k,j,s,0,z}^{(\varepsilon)}$  in Theorem 2 simplify to a single sum.

Apparently, the coefficients in Theorem 2 are not well defined in the (excluded) case  $k = 1$  and  $j = 0$ , as  $\Gamma(0)$  is involved in the numerator of  $\lambda_{n,1,0,s,i,z}^{(\varepsilon)}$ . Although this issue can be resolved by a proper interpretation of the (otherwise ambiguous) expression  $\Gamma(p) \cdot p = \Gamma(p+1)$  as 1 for  $p = 0$ , we prefer to state and derive this case separately. In fact, our analysis leads to substantial simplifications of the constants, as our next result shows.

**Theorem 3.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $i, r, s \in \mathbb{N}_0$ . Then*

$$\begin{aligned} \int_{A(n,1)} Q(E)^i \phi_{0,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2} + i)}{\pi\Gamma(\frac{n+s+1}{2} + i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z} Q^{\frac{s}{2}+i-z} \phi_{n-1}^{r,2z,0}(K, \beta) \end{aligned}$$

for even  $s$ . If  $s$  is odd, then

$$\int_{A(n,1)} Q(E)^i \phi_{0,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2} + i + 1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2} + i)} Q^{\frac{s-1}{2}+i} \phi_{n-1}^{r,1,0}(K, \beta).$$

Note that in Theorem 3 the Crofton integral is expressed only by tensorial curvature measures  $\phi_{n-1}^{r,z,0}$  (multiplied with suitable powers of the metric tensor), whereas generalized tensorial curvature measures are not needed. A global version of Theorem 3 is obtained by simply setting  $\beta = \mathbb{R}^n$ .

A translation invariant, global version of Theorem 2 allows us to combine several of the summands on the right-hand side of the formula.

**Theorem 4.** *Let  $K \in \mathcal{K}^n$  and  $i, j, k, s \in \mathbb{N}_0$  with  $j < k < n$  and  $k > 1$ . Then*

$$\int_{A(n,k)} Q(E)^i \Phi_{j,E}^s(K \cap E) \mu_k(dE) = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \Phi_{n-k+j}^{s+2i-2z}(K),$$

where  $\gamma_{n,k,j}$  and  $\lambda_{n,k,j,s,i,z}^{(0)}$  are defined as in Theorem 2, but

$$\vartheta_{n,k,j,s,i,z,p,q}^{(0)} := (n - k + j) \binom{\frac{k-1}{2} + p}{z} - (p(n - k) - q(k - 1)) \left(1 + \frac{k-j-1}{s+2i-2z-1} \left(1 - \frac{z}{p+q}\right)\right)$$

replaces  $\vartheta_{n,k,j,p,q}^{(0)}$ , except if  $s$  is odd and  $z = \lfloor \frac{s}{2} \rfloor + i$ , where  $\lambda_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i}^{(0)} := 0$ .

In Theorem 4, if  $p = q = 0$ , then the definition of  $\lambda_{n,k,j,s,i,z}^{(0)}$  implies that also  $z = 0$  and thus,  $\vartheta_{n,k,j,s,i,0,0,0}^{(0)}$  is well-defined with  $\frac{z}{p+q} = 1$ .

**3.2. Some Special Cases.** In the following, we restrict to the case  $i = 0$  of Crofton formulae for unweighted intrinsic Minkowski tensors or tensorial curvature measures.

**Corollary 1.** *Let  $K \in \mathcal{K}^n$  and  $k, j, s \in \mathbb{N}_0$  with  $0 \leq j < k < n$ . Then*

$$\int_{A(n,k)} \Phi_{j,E}^s(K \cap E) \mu_k(dE) = \delta_{n,k,j,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \eta_{n,k,j,s,z} Q^z \Phi_{n-k+j}^{s-2z}(K),$$



where

$$\begin{aligned}\delta_{n,k,j,s} &:= \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{k+1}{2})}{\pi\Gamma(\frac{n-k+j+s}{2}+1)}, \\ \eta_{n,k,j,s,z} &:= \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{j+s}{2} - q + 1) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \\ &\quad \times \left( \frac{n-k+j}{2} + q + \frac{(k-j-1)(q-z)}{s-2z-1} \right),\end{aligned}$$

but  $\eta_{n,k,j,s,\lfloor \frac{s}{2} \rfloor} := 0$  if  $s$  is odd.

### Specific choices of $s$

Next we collect some special cases of Corollary 1, which are obtained for specific choices of  $s \in \mathbb{N}_0$  by applications of Legendre's duplication formula and elementary calculations.

**Corollary 2.** *Let  $K \in \mathcal{K}^n$  and  $k, j \in \mathbb{N}_0$  with  $0 \leq j < k < n$ . Then*

$$\begin{aligned}\int_{A(n,k)} \Phi_{j,E}^2(K \cap E) \mu_k(dE) \\ = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+3}{2})\Gamma(\frac{j+1}{2})} \left( \frac{n-k}{4(n-k+j)} Q\Phi_{n-k+j}^0(K) + \frac{n-k+nj+j}{2(n-k+j)} \Phi_{n-k+j}^2(K) \right).\end{aligned}$$

**Corollary 3.** *Let  $K \in \mathcal{K}^n$  and  $k, j \in \mathbb{N}_0$  with  $0 \leq j < k < n$ . Then*

$$\int_{A(n,k)} \Phi_{j,E}^3(K \cap E) \mu_k(dE) = \frac{j+1}{n-k+j+1} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j}{2})} \Phi_{n-k+j}^3(K).$$

As  $\Gamma(\frac{j}{2})^{-1} = 0$ , for  $j = 0$ , the integral in Corollary 3 equals 0 in this case. However, as the integrand on the left-hand side is already 0, this is not surprising. The same is true for any odd number  $s \in \mathbb{N}$  and  $j = 0$ .

Corollary 3 immediately leads to a result which was obtained and applied by Bernig and Hug in [3, Lemma 4.13].

**Corollary 4.** *Let  $K \in \mathcal{K}^n$ . Then*

$$\int_{A(n,2)} \Phi_{1,E}^3(K \cap E) \mu_k(dE) = \binom{n}{2}^{-1} \Phi_{n-1}^3(K).$$

**The choice  $j = k - 1$**

Furthermore, we obtain simple Crofton formulae for the specific choice  $j = k - 1$  in the local and in the global case.

**Corollary 5.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $k, r, s \in \mathbb{N}_0$  with  $1 < k < n$ . Then*

$$\int_{A(n,k)} \phi_{k-1,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \delta_{n,k,k-1,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \xi_{n,k,s,z} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

where

$$\xi_{n,k,s,z} := \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma\left(q + \frac{1}{2}\right) \frac{\Gamma\left(\frac{k+s+1}{2} - q\right) \Gamma\left(\frac{n-k}{2} + q\right)}{\Gamma\left(\frac{n-1}{2} + q\right)}.$$

Corollary 5 will be derived from Theorem 2 in the same way as Theorem 4 is proved. More specifically, relation (4) is applied, which can be considered as a local version of Lemma 1 in the particular case  $j = n-1$ . Although  $k = 1$  is excluded in Corollary 5, the result is formally consistent with Theorem 3 (for  $i = 0$ ), which can be checked by simplifying the coefficients  $\xi_{n,1,s,z}$  with the help of Zeilberger's algorithm.

A global version of Corollary 5 is obtained by setting  $\beta = \mathbb{R}^n$ .

Finally, Theorem 3 can be globalized to give a result, which was obtained in [15] by a completely different approach.

**Corollary 6.** *Let  $K \in \mathcal{K}^n$  and  $s \in \mathbb{N}_0$ . Then*

$$\int_{A(n,1)} \Phi_{0,E}^s(K \cap E) \mu_k(dE) = \frac{2\omega_{n+s+1}}{\pi\omega_{s+1}\omega_n} \sum_{z=0}^{\frac{s}{2}} \frac{(-1)^z}{1-2z} \binom{\frac{s}{2}}{z} Q^{\frac{s}{2}-z} \Phi_{n-1}^{2z}(K)$$

for even  $s$ . For odd  $s$  the integral on the left-hand side equals 0.

Note that if  $s \in \mathbb{N}$  is odd, then the Crofton integral in Theorem 3 is a non-zero measure, as the tensorial curvature measures  $\phi_{n-1}^{r,1,0}(K, \cdot)$  are non-zero (if the underlying set  $K$  is at least  $(n-1)$ -dimensional), whereas  $\Phi_{n-1}^1 \equiv 0$  in the global case considered in Corollary 6.

**3.3. Crofton Formulae for Extrinsic Tensorial Curvature Measures.** In the following, we state Crofton formulae for tensorial curvature measures for  $j = k-1$ . The method also applies to the cases where  $j \leq k-2$ , but it remains to be explored to which extent the constants can be simplified then. As for the intrinsic versions, we have to distinguish between the cases  $k > 1$  and  $k = 1$ . We start with the former.

**Theorem 5.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $k, r, s \in \mathbb{N}_0$  with  $1 < k < n$ . Then*

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

where

$$\kappa_{n,k,s,z} := \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2}) \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2} + z) \Gamma(\frac{k+s-1}{2} - z)}{\Gamma(\frac{s}{2} - z + 1) z!}$$

if  $z \neq \frac{s-1}{2}$ , and

$$(8) \quad \kappa_{n,k,s,\frac{s-1}{2}} := \pi^{\frac{n-k-1}{2}} \frac{2k(n+s-2)}{(n-1)(n-k+s-1)} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n+s+1}{2})}.$$

In Theorem 5, if  $s$  is odd the coefficient  $\kappa_{n,k,s,(s-1)/2}$  has to be defined separately, as the proof shows. (In fact, the difference amounts to a factor  $k(n+s-2)[(k-1)(n+s-1)]^{-1}$ .) For even  $s$ , the constants involved in the proof of Theorem 5 can be simplified by a direct calculation to arrive at the asserted result. However, if  $s$  is odd, we need the connection to the work [3] to simplify the constants. Since this connection breaks down for  $z = (s-1)/2$ ,  $s$  odd, a separate direct calculation is required for this case, and this finally yields the correct constant in (8). The result is also consistent with the special case  $k = 1$  which is considered next.

For  $k = 1$  the Crofton integrals can be represented with a single functional, as the following theorem shows.

**Theorem 6.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $r, s \in \mathbb{N}_0$ . Then*

$$\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) = \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\lfloor \frac{s+1}{2} \rfloor + \frac{1}{2})}{\Gamma(\frac{n}{2} + \lfloor \frac{s+1}{2} \rfloor)} Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta).$$

It can be easily checked that the result for  $k = 1$  can be obtained from the one for  $k > 1$  by a formal specialization and proper interpretation of expressions which a priori are not well defined. For this to work, it is indeed crucial that for odd values of  $s$  and  $z = (s-1)/2$  the definition in (8) applies.

In [3, Proposition 4.10], an alternative basis of the vector space of continuous, translation invariant and rotation covariant  $\mathbb{T}^p$ -valued valuations on  $\mathcal{K}^n$  was introduced, based on the trace free part of the Minkowski tensors, which was called the  $\Psi$ -basis. In the same spirit (but locally and with the current normalization), we now define

$$\psi_k^{r,s,0} := \phi_k^{r,s,0} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + s - 1)} Q^j \phi_k^{r,s-2j,0}$$

for  $r, s \in \mathbb{N}_0$  and  $k \in \{0, \dots, n-1\}$ . Interpreting this definition in the right way if  $n = 2$  and  $s = 0$  (where  $\psi_k^{r,0,0} = \phi_k^{r,0,0}$ ), we can also write

$$(9) \quad \psi_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + s - 1)} Q^j \phi_k^{r,s-2j,0}.$$

In particular,  $\psi_k^{r,s,0} = \phi_k^{r,s,0}$  for  $s \in \{0, 1\}$ . Conversely, we have

$$(10) \quad \phi_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - 2j)}{\Gamma(\frac{n}{2} + s - j)} Q^j \psi_k^{r,s-2j,0}.$$

Although this will not be needed explicit, it shows how we can switch between a  $\phi$ -representation and a  $\psi$ -representation of tensorial curvature measures.

The main advantage of the new local tensor valuations given in (9) is that the Crofton formula takes a particularly simple form.

**Corollary 7.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and let  $k, r, s \in \mathbb{N}_0$  with  $0 < k < n$ . Then*

$$\begin{aligned} & \int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \pi^{\frac{n-k}{2}} \frac{k-1}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n-k+s+1}{2})} \psi_{n-1}^{r,s,0}(K, \beta). \end{aligned}$$

For  $r = 0$  and  $\beta = \mathbb{R}^n$ , Corollary 7 coincides with [3, Corollary 6.1] (in the case corresponding to  $j = k - 1$ ). If  $s \in \{0, 1\}$ , then  $\psi_k^{r,s,0} = \phi_k^{r,s,0}$  and Corollary 7 coincides with Theorem 5 (resp. Theorem 6, for  $k = 1$ ). If  $k = 1$ , then the integral in Corollary 7 vanishes, except for  $s \in \{0, 1\}$ .

#### 4. PROOFS OF THE MAIN RESULTS

In this section, we first recall some results from [12]. Then we prove an integral formula which is required in the following. Finally, all ingredients are combined for the proofs of our main theorems.

A basic tool is the following transformation formula (see [12, Corollary 4.2]). It can be used to carry out an integration over linear Grassmann spaces recursively. The result is also true for  $k = 1$ , but in this case the outer integration on the right-hand side is trivial.

**Lemma 2.** *Let  $u \in \mathbb{S}^{n-1}$  and let  $h : G(n, k) \rightarrow \mathbb{T}^p$  be an integrable function for  $k, p \in \mathbb{N}_0$ ,  $0 < k < n$ . Then*

$$\begin{aligned} \int_{G(n,k)} h(L) \nu_k(dL) &= \frac{\omega_k}{2\omega_n} \int_{G(u^\perp, k-1)} \int_{-1}^1 \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1} (1-t^2)^{\frac{n-k-2}{2}} \\ &\quad \times h(\text{span}\{U, tu + \sqrt{1-t^2}w\}) \mathcal{H}^{n-k-1}(dw) dt \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

The next results are derived from the previous one (see [12, Lemma 4.3 and Corollary 4.6]).

**Lemma 3.** *Let  $i, k \in \mathbb{N}_0$  with  $k \leq n$ . Then*

$$\int_{G(n,k)} Q(L)^i \nu_k(dL) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i)\Gamma(\frac{k}{2})} Q^i.$$

In Lemma 3, we interpret the coefficient of the tensor on the right-hand side as 0, if  $k = 0$  and  $i \neq 0$ , and as 1, if  $k = i = 0$ , as  $\Gamma(0)^{-1} := 0$  and  $\frac{\Gamma(a)}{\Gamma(a)} = 1$  for all  $a \in \mathbb{R}$ .

**Lemma 4.** *Let  $i \in \mathbb{N}_0$ ,  $k, r \in \{0, \dots, n\}$  with  $k + r \geq n$ , and let  $F \in G(n, r)$ . Then*

$$\begin{aligned} \int_{G(n,k)} [F, L]^2 Q(L)^i \nu_k(dL) &= \frac{r!k!}{n!(k+r-n)!} \frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i+1)\Gamma(\frac{k}{2}+1)} \\ &\quad \times ((\frac{k}{2}+i)Q^i + i\frac{k-n}{r}Q^{i-1}Q(F)). \end{aligned}$$

We interpret the second summand on the right-hand side of Lemma 4 as 0, if  $i = 0$ , which is consistent with [12, Lemma 4.4]. If  $r = 0$ , we also interpret the second summand as 0 and the integral on the left equals  $Q^i$ .

Finally, we state the following integral formula (see [12, p. 503]), which is a special case of [17, Theorem 3.1].

**Lemma 5.** *Let  $P \in \mathcal{P}^n$  be a polytope,  $L \in G(n, k)$  for  $0 \leq j < k < n$  and let  $g : \mathbb{R}^n \times (\mathbb{S}^{n-1} \cap L) \rightarrow \mathbb{T}$  be a measurable bounded function. Then*

$$\begin{aligned} & \int_{L^\perp} \int_{L_t \times (L \cap \mathbb{S}^{n-1})} g(x, u) \Lambda_j^{(L_t)}(P \cap L_t, d(x, u)) \mathcal{H}^{n-k}(dt) \\ &= \frac{1}{\omega_{k-j}} \sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \times (N(P, F) \cap \mathbb{S}^{n-1})} g(x, \pi_L(u)) \|p_L(u)\|^{j-k} [F, L]^2 \mathcal{H}^{n-1}(d(x, u)). \end{aligned}$$

**4.1. Auxiliary Integral Formulae.** With the preliminary results from [12] we are able to establish the following integral formula, which is a slightly modified version of [12, Proposition 4.7].

**Proposition 1.** *Let  $i, j, k, s \in \mathbb{N}_0$  with  $j < k < n$  and  $k > 1$ ,  $F \in G(n, n-k+j)$  and  $u \in F^\perp \cap \mathbb{S}^{n-1}$ . Then*

$$\begin{aligned} & \int_{G(n, k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \gamma_{n, k, j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} (\lambda_{n, k, j, s, i, z}^{(0)} u^2 + \lambda_{n, k, j, s, i, z}^{(1)} Q(F)) Q^z u^{s+2i-2z-2}, \end{aligned}$$

where the coefficients are defined as in Theorem 2.

*Proof.* Lemma 2 yields

$$\begin{aligned} & \int_{G(n, k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\omega_k}{2\omega_n} \int_{G(u^\perp, k-1)} \int_{-1}^1 \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1} (1-t^2)^{\frac{n-k-2}{2}} \pi_{\text{span}\{U, tu + \sqrt{1-t^2}w\}}(u)^s \\ & \quad \times Q(\text{span}\{U, tu + \sqrt{1-t^2}w\})^i \|p_{\text{span}\{U, tu + \sqrt{1-t^2}w\}}(u)\|^{j-k} \\ & \quad \times [F, \text{span}\{U, tu + \sqrt{1-t^2}w\}]^2 \mathcal{H}^{n-k-1}(dw) dt \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

As

$$\begin{aligned} Q(\text{span}\{U, tu + \sqrt{1-t^2}w\}) &= Q(U) + (|t|u + \sqrt{1-t^2} \text{sign}(t)w)^2, \\ \pi_{\text{span}\{U, tu + \sqrt{1-t^2}w\}}(u) &= |t|u + \sqrt{1-t^2} \text{sign}(t)w, \\ \|p_{\text{span}\{U, tu + \sqrt{1-t^2}w\}}(u)\| &= |t|, \\ [F, \text{span}\{U, tu + \sqrt{1-t^2}w\}] &= [F, U]^{(u^\perp)} |t| \end{aligned}$$

hold for all  $t \in [-1, 1] \setminus \{0\}$ , we obtain

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\omega_k}{2\omega_n} \int_{G(u^\perp, k-1)} \int_{-1}^1 \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{j+1} (1-t^2)^{\frac{n-k-2}{2}} ([F, U]^{(u^\perp)})^2 (|t|u + \sqrt{1-t^2}w)^s \\ & \quad \times (Q(U) + (|t|u + \sqrt{1-t^2}w)^2)^i \mathcal{H}^{n-k-1}(dw) dt \nu_{k-1}^{u^\perp}(dU), \end{aligned}$$

where we used the fact that the integration with respect to  $w$  is invariant under reflections in the origin. Then we apply the binomial theorem to the terms  $(Q(U) + (|t|u + \sqrt{1-t^2}w)^2)^i$  and  $(|t|u + \sqrt{1-t^2}w)^{s+2p}$  and get

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\omega_k}{2\omega_n} \sum_{p=0}^i \sum_{q=0}^{s+2p} \binom{i}{p} \binom{s+2p}{q} \int_{G(u^\perp, k-1)} \int_{-1}^1 |t|^{j+s+2p-q+1} (1-t^2)^{\frac{n-k+q-2}{2}} dt \\ & \quad \times \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \mathcal{H}^{n-k-1}(dw) ([F, U]^{(u^\perp)})^2 u^{s+2p-q} Q(U)^{i-p} \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

Since

$$\int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \mathcal{H}^{n-k-1}(dw) = \mathbf{1}\{q \text{ even}\} 2 \frac{\omega_{n-k+q}}{\omega_{q+1}} Q(U^\perp \cap u^\perp)^{\frac{q}{2}},$$

we deduce from the definition of the Beta function and its relation to the Gamma function that

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\omega_k}{\omega_n} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \frac{\Gamma(\frac{j+s}{2} + p - q + 1) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + p + 1)} \frac{\omega_{n-k+2q}}{\omega_{2q+1}} \\ & \quad \times u^{s+2p-2q} \int_{G(u^\perp, k-1)} Q(U^\perp \cap u^\perp)^q ([F, U]^{(u^\perp)})^2 Q(U)^{i-p} \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

Applying the binomial theorem to  $Q(U^\perp \cap u^\perp)^q = (Q(u^\perp) - Q(U))^q$  yields

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{k}{2})} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^q (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{j+s}{2} + p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + p + 1)} \\ (11) \quad & \times u^{s+2p-2q} Q(u^\perp)^{q-y} \int_{G(u^\perp, k-1)} ([F, U]^{(u^\perp)})^2 Q(U)^{i-p+y} \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

We conclude from Lemma 4, which is applied in  $u^\perp$  to the remaining integral on the right-hand side of (11),

$$\begin{aligned}
& \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\
&= \frac{(n-k+j)!(k-1)! \Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2})}{\sqrt{\pi}(n-1)!j! \Gamma(\frac{k}{2}) \Gamma(\frac{k+1}{2})} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \Gamma(q + \frac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{j+s}{2} + p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + p + 1)} u^{s+2p-2q} \sum_{y=0}^q (-1)^y \binom{q}{y} \frac{\Gamma(\frac{k-1}{2} + i - p + y)}{\Gamma(\frac{n+1}{2} + i - p + y)} \\
&\quad \times \left( \left( \frac{k-1}{2} + i - p + y \right) Q(u^\perp)^{i-p+q} + \frac{k-n}{n-k+j} (i - p + y) Q(u^\perp)^{i-p+q-1} Q(F) \right).
\end{aligned}$$

Lemma 7 from Section 6 applied twice to the summations with respect to  $y$  and Legendre's duplication formula applied three times to the Gamma functions involving  $n$ ,  $k$  and  $n-k$  yield together with the definitions of  $\gamma_{n,k,j}$  and  $\vartheta_{n,k,j,p,q}^{(\varepsilon)}$ ,  $\varepsilon \in \{0, 1\}$ ,

$$\begin{aligned}
& \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\
&= \gamma_{n,k,j} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q + \frac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)} \frac{\Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + p + q)} \\
&\quad \times u^{s+2i-2p-2q} \left( \vartheta_{n,k,j,p,q}^{(0)} Q(u^\perp)^{p+q} - \vartheta_{n,k,j,p,q}^{(1)} Q(u^\perp)^{p+q-1} Q(F) \right),
\end{aligned}$$

where we changed the order of summation with respect to  $p$ . From the binomial theorem applied to  $Q(u^\perp)^{p+q} = (Q - u^2)^{p+q}$  we obtain

$$\begin{aligned}
& \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\
&= \gamma_{n,k,j} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)} \\
&\quad \times \frac{\Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + p + q)} \left( \sum_{z=0}^{p+q} (-1)^{p+q-z} \binom{p+q}{z} \vartheta_{n,k,j,p,q}^{(0)} Q^z u^{s+2i-2z} \right. \\
&\quad \left. + \sum_{z=0}^{p+q-1} (-1)^{p+q-z} \binom{p+q-1}{z} \vartheta_{n,k,j,p,q}^{(1)} Q^z u^{s+2i-2z-2} Q(F) \right).
\end{aligned}$$

A change of the order of summation, such that we sum with respect to  $z$  first, gives

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} (\lambda_{n,k,j,s,i,z}^{(0)} u^2 + \lambda_{n,k,j,s,i,z}^{(1)} Q(F)) Q^z u^{s+2i-2z-2}, \end{aligned}$$

which concludes the proof.  $\square$

Next we state the special case of Proposition 1 where  $k = 1$ .

**Proposition 2.** *Let  $i, s \in \mathbb{N}_0$ ,  $F \in G(n, n-1)$  and  $u \in F^\perp \cap \mathbb{S}^{n-1}$ . Then*

$$\begin{aligned} & \int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F, L]^2 \nu_1(dL) \\ &= \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s+1}{2} + i)}{\pi \Gamma(\frac{n+s+1}{2} + i)} \sum_{z=0}^{\frac{s}{2} + i} (-1)^z \binom{\frac{s}{2} + i}{z} \frac{1}{1-2z} u^{2z} Q^{\frac{s}{2} + i - z} \end{aligned}$$

for even  $s$ . If  $s$  is odd, then

$$\int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F, L]^2 \nu_k(dL) = \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s}{2} + i + 1)}{\sqrt{\pi} \Gamma(\frac{n+s+1}{2} + i)} u Q^{\frac{s-1}{2} + i}.$$

*Proof.* The proof basically works as the proof of Proposition 1. But we do not need to apply Lemma 4 as (11) simplifies to

$$\begin{aligned} & \int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F, L]^2 \nu_k(dL) \\ &= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{p=0}^i \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^q (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{s}{2} + p - q + 1)}{\Gamma(\frac{n+s+1}{2} + p)} \\ & \quad \times u^{s+2p-2q} Q(u^\perp)^{q-y} \int_{G(u^\perp, 0)} ([F, U]^{(u^\perp)})^2 Q(U)^{i-p+y} \nu_{k-1}^{u^\perp}(dU). \end{aligned}$$

Since the remaining integral on the right-hand side equals 1, if  $p = i$  and  $y = 0$ , and in all the other cases it equals 0, we obtain

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \binom{s+2i}{2q} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{s}{2} + i - q + 1)}{\Gamma(\frac{n+s+1}{2} + i)} u^{s+2i-2q} Q(u^\perp)^q. \end{aligned}$$



Applying the binomial theorem to  $Q(u^\perp)^q = (Q - u^2)^q$  yields

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \sum_{z=0}^q (-1)^{q-z} \binom{s+2i}{2q} \binom{q}{z} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{s}{2} + i - q + 1)}{\Gamma(\frac{n+s+1}{2} + i)} u^{s+2i-2z} Q^z. \end{aligned}$$

A change of the order of summation and Legendre's duplication formula applied to the Gamma functions involving  $q$  give

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{(s+2i)! \Gamma(\frac{n}{2})}{2^{s+2i} \Gamma(\frac{n+s+1}{2} + i)} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \frac{1}{z!} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2} + i - q)(q-z)!} u^{s+2i-2z} Q^z. \end{aligned}$$

If  $s$  is even, we conclude from Lemma 8 applied to the summation with respect to  $q$  and from another application of Legendre's duplication formula that

$$\begin{aligned} & \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \\ &= \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s+1}{2} + i)}{\pi \Gamma(\frac{n+s+1}{2} + i)} \sum_{z=0}^{\frac{s}{2} + i} (-1)^{\frac{s}{2} + i - z + 1} \binom{\frac{s}{2} + i}{z} \frac{1}{s+2i-2z-1} u^{s+2i-2z} Q^z. \end{aligned}$$

A change of the order of summation with respect to  $z$  then yields the assertion.

On the other hand, if  $s$  is odd, the binomial theorem gives, for  $\lfloor \frac{s}{2} \rfloor + i \neq z$ ,

$$\begin{aligned} & \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2} + i - q)(q-z)!} = \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - z} (-1)^q \binom{\lfloor \frac{s}{2} \rfloor + i - z}{q} \\ &= \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} (1-1)^{\lfloor \frac{s}{2} \rfloor + i - z} \\ &= 0. \end{aligned} \tag{12}$$

For  $\lfloor \frac{s}{2} \rfloor + i = z$ , the sum on the left-hand side of (12) equals 1. Hence, we finally obtain

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) = \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s}{2} + i + 1)}{\sqrt{\pi} \Gamma(\frac{n+s+1}{2} + i)} u Q^{\lfloor \frac{s}{2} \rfloor + i},$$

if  $s$  is odd. □

**4.2. The Proofs for the Intrinsic Case.** Now all tools are available which are needed to prove the main theorems.

We start with the proof of Theorem 1.

*Proof (Theorem 1).* Let  $L \in G(n, k)$  and  $t \in L^\perp$ . Then we have

$$\phi_{k, L_t}^{r, s, 0}(K \cap L_t, \beta \cap L_t) = \mathbf{1}\{s = 0\} \int_{K \cap \beta \cap L_t} x^r \mathcal{H}^k(dx)$$

and thus, for  $s \neq 0$ ,

$$\begin{aligned} & \int_{A(n, k)} Q(E)^i \phi_{k, E}^{r, s, 0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \int_{G(n, k)} \int_{L^\perp} Q(L_t)^i \phi_{k, L_t}^{r, s, 0}(K \cap L_t, \beta \cap L_t) \mathcal{H}^{n-k}(dt) \nu_k(dL) = 0. \end{aligned}$$

Furthermore, for  $s = 0$  Fubini's theorem yields

$$\begin{aligned} & \int_{A(n, k)} Q(E)^i \phi_{k, E}^{r, 0, 0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \int_{G(n, k)} Q(L)^i \int_{L^\perp} \int_{K \cap \beta \cap L_t} x^r \mathcal{H}^k(dx) \mathcal{H}^{n-k}(dt) \nu_k(dL) \\ &= \int_{G(n, k)} Q(L)^i \nu_k(dL) \int_{K \cap \beta} x^r \mathcal{H}^n(dx). \end{aligned}$$

Then we conclude the proof with Lemma 3 and the definition of  $\phi_n^{r, 0, 0}$ . □

We turn to the proof of Theorem 2.

*Proof (Theorem 2).* First, we prove the formula for a polytope  $P \in \mathcal{P}^n$ . The general result then follows by an approximation argument.

As a matter of convenience, we name the integral of interest  $I$ . Then Lemma 5 yields

$$\begin{aligned} I &= \omega_{k-j} \int_{G(n, k)} Q(L)^i \int_{L^\perp} \int_{L_t \times (L \cap \mathbb{S}^{n-1})} \mathbf{1}_\beta(x) x^r u^s \Lambda_j^{(L_t)}(P \cap L_t, d(x, u)) \mathcal{H}^{n-k}(dt) \nu_k(dL) \\ &= \sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(dx) \int_{G(n, k)} Q(L)^i \\ &\quad \times \int_{N(P, F) \cap \mathbb{S}^{n-1}} \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \mathcal{H}^{k-j-1}(du) \nu_k(dL). \end{aligned}$$

With Fubini's theorem we conclude

$$\begin{aligned} I &= \sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(dx) \int_{N(P, F) \cap \mathbb{S}^{n-1}} \\ (13) \quad & \times \int_{G(n, k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \nu_k(dL) \mathcal{H}^{k-j-1}(du). \end{aligned}$$

Then we obtain from Proposition 1

$$\begin{aligned}
I &= \gamma_{n,k,j} \sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(dx) \\
&\times \left( \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z} \mathcal{H}^{k-j-1}(du) \right. \\
&\quad \left. + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z Q(F) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z-2} \mathcal{H}^{k-j-1}(du) \right).
\end{aligned}$$

With the definition of the tensorial curvature measures we get

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \phi_{n-k+j}^{r,s+2i-2z,0}(P, \beta) + \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \phi_{n-k+j}^{r,s+2i-2z-2,1}(P, \beta).$$

Combining the two sums yields the assertion in the polytopal case.

As pointed out before, there exists a weakly continuous extension of the generalized tensorial curvature measures  $\phi_{n-k+j}^{r,s+2i-2z-2,1}$  from the set of all polytopes to  $\mathcal{K}^n$ . The same is true for the tensorial curvature measures  $\phi_{n-k+j}^{r,s+2i-2z,0}$ . Hence, approximating a convex body  $K \in \mathcal{K}^n$  by polytopes yields the assertion in the general case.  $\square$

Now we prove Theorem 3, which deals with the case  $k = 1$  excluded in the statement of Theorem 2.

*Proof (Theorem 3).* The proof basically works as the one of Theorem 2. Again, we prove the formula for a polytope  $P \in \mathcal{P}^n$ . We call the integral of interest  $I$  and proceed as in the previous proof in order to obtain (13). Now we apply Proposition 2 and obtain

$$\begin{aligned}
I &= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2} + i)}{\pi\Gamma(\frac{n+s+1}{2} + i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2} + i}{z} \frac{1}{1-2z} Q^{\frac{s}{2}+i-z} \\
&\times \sum_{F \in \mathcal{F}_{n-1}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(dx) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{2z} \mathcal{H}^0(du),
\end{aligned}$$

if  $s$  is even. Hence, we conclude the assertion with the definition of  $\phi_{n-1}^{r,2z,0}$ .

If  $s$  is odd, Proposition 2 yields

$$I = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2} + i + 1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2} + i)} Q^{\frac{s-1}{2}+i} \phi_{n-1}^{r,1,0}(P, \beta).$$

As sketched in the proof of Theorem 2, the general result follows by an approximation argument.  $\square$

For the proof of Theorem 4, we first globalize Theorem 2 and then apply Lemma 1 to treat the appearing tensors  $\phi_{n-k+j}^{0,s+2i-2z-2,1}$ .

*Proof (Theorem 4).* We only prove the formula for a polytope  $P \in \mathcal{P}^n$ . As before, the general result follows by an approximation argument.

We briefly write  $I$  for the Crofton integral under investigation. Starting from the special case of Theorem 2 where  $r = 0$  and  $\beta = \mathbb{R}^n$ , we obtain

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \Phi_{n-k+j}^{s+2i-2z}(P) + \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \\ \times \sum_{F \in \mathcal{F}_{n-k+j}(P)} Q(F) \mathcal{H}^{n-k+j}(F) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z-2} \mathcal{H}^{k-j-1}(du).$$

With  $Q(F) = Q - Q(N(P, F))$  and Lemma 1 we get

$$\sum_{F \in \mathcal{F}_{n-k+j}(P)} Q(F) \mathcal{H}^{n-k+j}(F) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z-2} \mathcal{H}^{k-j-1}(du) \\ = Q \Phi_{n-k+j}^{s+2i-2z-2}(P) - \frac{k-j+s+2i-2z-2}{s+2i-2z-1} \Phi_{n-k+j}^{s+2i-2z}(P)$$

and thus

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \Phi_{n-k+j}^{s+2i-2z}(P) \\ + \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \left( Q \Phi_{n-k+j}^{s+2i-2z-2}(P) - \frac{k-j+s+2i-2z-2}{s+2i-2z-1} \Phi_{n-k+j}^{s+2i-2z}(P) \right).$$

Combining these sums yields

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \left( \lambda_{n,k,j,s,i,z}^{(0)} + \lambda_{n,k,j,s,i,z-1}^{(1)} - \frac{k-j+s+2i-2z-2}{s+2i-2z-1} \lambda_{n,k,j,s,i,z}^{(1)} \right) Q^z \Phi_{n-k+j}^{s+2i-2z}(P).$$

In fact, we have  $\lambda_{n,k,j,s,i,-1}^{(1)} = 0$  and, furthermore for even  $s$ , as the sum with respect to  $q$  is empty,  $\lambda_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i}^{(1)}$  also vanishes. On the other hand, for odd  $s$ , as  $\Phi_{n-k+j}^1 \equiv 0$ , the last summand of the sum with respect to  $z$  actually vanishes and thus its coefficient does not have to be determined and is defined as zero.

Hence, we obtained a representation of the integral with the desired Minkowski tensors. It remains to determine the coefficients explicitly. First, we consider the case where  $(k > 1)$  and

$z \in \{1, \dots, \lfloor \frac{s}{2} \rfloor + i - 1\}$ . We get

$$\begin{aligned} & \lambda_{n,k,j,s,i,z}^{(0)} + \lambda_{n,k,j,s,i,z-1}^{(1)} \\ &= \sum_{p=0}^i \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\ & \quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1) \Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1) \Gamma(\frac{n+1}{2} + p + q)} \\ & \quad \times \left( (n-k+j)(\frac{k-1}{2} + p) - \frac{z}{p+q} (p(n-k) - q(k-1)) \right) \end{aligned}$$

and

$$\begin{aligned} \lambda_{n,k,j,s,i,z}^{(1)} &= \sum_{p=0}^i \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\ & \quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1) \Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1) \Gamma(\frac{n+1}{2} + p + q)} \\ (14) \quad & \quad \times \frac{p+q-z}{p+q} (p(n-k) - q(k-1)). \end{aligned}$$

Hence we conclude

$$\begin{aligned} & \lambda_{n,k,j,s,i,z}^{(0)} + \lambda_{n,k,j,s,i,z-1}^{(1)} - \frac{k-j+s+2i-2z-2}{s+2i-2z-1} \lambda_{n,k,j,s,i,z}^{(1)} \\ &= \sum_{p=0}^i \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\ & \quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1) \Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1) \Gamma(\frac{n+1}{2} + p + q)} \\ & \quad \times \left( (n-k+j)(\frac{k-1}{2} + p) - \frac{p(n-k)-q(k-1)}{p+q} \left( p+q + \frac{(k-j-1)(p+q-z)}{s+2i-2z-1} \right) \right). \end{aligned}$$

The case  $z = \lfloor \frac{s}{2} \rfloor + i$ , for even  $s$ , follows similarly. For  $z = 0$ , we have  $\lambda_{n,k,j,s,i,-1}^{(1)} = 0$  and (14) still holds, if one cancels the remaining  $\frac{p+q-z}{p+q} = 1$ .  $\square$

Finally, we provide the argument for Corollary 5, which is the special case of Theorem 2 obtained for  $i = 0$  and  $j + 1 = k \geq 2$ .

*Proof (Corollary 5).* With the specific choices of the indices, we obtain

$$\begin{aligned} \lambda_{n,k,k-1,s,0,z}^{(\varepsilon)} &= \sum_{q=z+\varepsilon}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q-\varepsilon}{z} \Gamma(q + \tfrac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{k+s+1}{2} - q)}{\Gamma(\frac{n+s+1}{2})} \frac{\Gamma(\frac{k-1}{2}) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \vartheta_{n,k,k-1,0,q}^{(\varepsilon)}, \end{aligned}$$

with

$$\vartheta_{n,k,k-1,0,q}^{(0)} = \tfrac{1}{2}(n-1)(k-1), \quad \vartheta_{n,k,k-1,0,q}^{(1)} := -q(k-1),$$

and

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}.$$

Let us denote the Crofton integral by  $I$ . Then Theorem 2 implies that

$$\begin{aligned} I &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z (\lambda_{n,k,k-1,s,0,z}^{(0)} - \lambda_{n,k,k-1,s,0,z}^{(1)}) \phi_{n-1}^{r,s-2z,0}(K, \beta) \\ &\quad + \gamma_{n,k,k-1} \sum_{z=1}^{\lfloor \frac{s}{2} \rfloor + 1} Q^z \lambda_{n,k,k-1,s,0,z-1}^{(1)} \phi_{n-1}^{r,s-2z,0}(K, \beta) \\ &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z \underbrace{(\lambda_{n,k,k-1,s,0,z}^{(0)} + \lambda_{n,k,k-1,s,0,z-1}^{(1)} - \lambda_{n,k,k-1,s,0,z}^{(1)})}_{=: \lambda} \phi_{n-1}^{r,s-2z,0}(K, \beta), \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \\ &\quad \times \left[ \binom{q}{z} \frac{1}{2}(n-1)(k-1) - \binom{q-1}{z-1} (-1)q(k-1) - \binom{q-1}{z} (-1)q(k-1) \right] \\ &= \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \binom{q}{z} (k-1) \left( \frac{n-1}{2} + q \right) \\ &= 2 \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)}, \end{aligned}$$

from which the assertion follows.  $\square$

## 5. THE PROOFS FOR THE EXTRINSIC CASE

Our starting point is a relation, due to McMullen, which relates the intrinsic and the extrinsic Minkowski tensors (see [16, 5.1 Theorem]). Its proof can easily be localized (see [21, Korollar 2.2.2]). Combining this localization with the relation  $Q = Q(E) + Q(E^\perp)$ , where  $E \subset \mathbb{R}^n$  is any  $k$ -flat, we obtain the following lemma.

**Lemma 6.** *Let  $j, k, r, s \in \mathbb{N}_0$  with  $j < k < n$ , let  $K \in \mathcal{K}^n$  with  $K \subset E \in A(n, k)$  and  $\beta \in \mathcal{B}(\mathbb{R}^n)$ . Then*

$$\phi_j^{r,s,0}(K, \beta) = \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-j+s}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{k-j+s}{2} - m)}{4^m m! (s-2m)!} Q^l Q(E)^{m-l} \phi_{j,E}^{r,s-2m,0}(K, \beta).$$

We start with the proof of Theorem 5, for which we use Theorem 2 after an application of Lemma 6.

*Proof (Theorem 5).* Lemma 6 for  $j = k - 1$  gives

$$\begin{aligned} & \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^m (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} Q^l \\ & \quad \times \int_{A(n,k)} Q(E)^{m-l} \phi_{k-1,E}^{r,s-2m,0}(K \cap E, \beta \cap E) \mu_k(dE). \end{aligned}$$

For  $j = k - 1$  we can argue as in the proof of Corollary 5 to see that Theorem 2 implies that

$$\begin{aligned} & \int_{A(n,k)} Q(E)^i \phi_{k-1,E}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ (15) \quad &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,k-1,s,i,z} Q^z \phi_{n-1}^{r,s+2i-2z,0}(K \cap E, \beta \cap E), \end{aligned}$$

where

$$\begin{aligned} \lambda_{n,k,k-1,s,i,z} &= (k-1) \sum_{p=0}^i \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \\ & \quad \times \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} + i - p - q)}{\Gamma(\frac{n+s+1}{2} + i - p)} \frac{\Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + p + q)}. \end{aligned}$$

(Of course, for  $i = 0$  we recover Corollary 5.) Hence, we obtain

$$\begin{aligned} & \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^m \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - l} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} \\ & \quad \times \lambda_{n,k,k-1,s-2m,m-l,z} Q^{l+z} \phi_{n-1}^{r,s-2l-2z,0}(K, \beta). \end{aligned}$$

An index shift of the summation with respect to  $z$  yields

$$\begin{aligned} & \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^m \sum_{z=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} \\ & \quad \times \lambda_{n,k,k-1,s-2m,m-l,z-l} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta). \end{aligned}$$

Changing the order of summation gives

$$\begin{aligned} & \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^z \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} \\ (16) \quad & \quad \times \lambda_{n,k,k-1,s-2m,m-l,z-l} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta). \end{aligned}$$

The coefficients of the tensorial curvature measures on the right-hand side of (16) do not depend on the choice of  $r \in \mathbb{N}_0$  or  $\beta \in \mathcal{B}(\mathbb{R}^n)$ . Thus, we can set

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

where the coefficient  $\kappa_{n,k,s,z}$  is uniquely defined in the obvious way. By choosing  $r = 0$  and  $\beta = \mathbb{R}^n$ , we can compare this to the Crofton formula for translation invariant Minkowski tensors in [3]. In fact, since the functionals  $Q^z \phi_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n)$ ,  $z \in \{0, \dots, \lfloor s/2 \rfloor\} \setminus \{(s-1)/2\}$ , are linearly independent, we can conclude from the Crofton formula for the translation invariant Minkowski tensors in [3, Theorem 3] that

$$\kappa_{n,k,s,z} = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2}) \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2} + z) \Gamma(\frac{k+s-1}{2} - z)}{\Gamma(\frac{s}{2} - z + 1) z!}$$

for  $z \neq (s-1)/2$ . If  $z = (s-1)/2$ , then  $\phi_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n) = \Phi_{n-1}^1(K) = 0$ , and hence we do not get any information about the corresponding coefficient from the global theorem. Consequently, we have to calculate  $\kappa_{n,k,s,(s-1)/2}$  directly, which is what we do later in the proof.



But first we demonstrate that the coefficients of the tensorial curvature measures in (16) can be determined also by a direct calculation if  $s$  is even. In fact, we obtain

$$\begin{aligned}
S &:= \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m!(s-2m)!} \binom{m}{l} \lambda_{n,k,k-1,s-2m,m-l,z-l} \\
&= (k-1) \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} \sum_{p=l}^m \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^{m+l+p+q-z} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m!(s-2m)!} \\
&\quad \times \binom{m}{l} \binom{m-l}{p-l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q + \frac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{k+s+1}{2} - p - q)}{\Gamma(\frac{n+s+1}{2} - p)} \frac{\Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + p + q - l)}.
\end{aligned}$$

Changing the order of summation gives

$$\begin{aligned}
S &= (k-1) \sum_{p=l}^{\lfloor \frac{s}{2} \rfloor} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^{l+q-z} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q + \frac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{k+s+1}{2} - p - q)}{\Gamma(\frac{n+s+1}{2} - p)} \frac{\Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + p + q - l)} \\
&\quad \times \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \binom{m}{l} \binom{m-l}{p-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m!(s-2m)!}.
\end{aligned}$$

We denote the sum with respect to  $m$  by  $T$  and conclude

$$\begin{aligned}
T &= \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \binom{m}{l} \binom{m-l}{p-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m!(s-2m)!} \\
&= \frac{1}{l!(p-l)!} \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m (m-p)!(s-2m)!}.
\end{aligned}$$

An index shift yields

$$T = \frac{1}{2^s l!(p-l)!} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^m \frac{2^{s-2p-2m} \Gamma(\frac{s+1}{2} - p - m)}{m!(s-2p-2m)!}.$$

Legendre's duplication formula gives

$$T = \frac{\sqrt{\pi}}{2^s l!(p-l)!} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - p - m + 1)}.$$

If  $s$  is even, the binomial theorem yields

$$\begin{aligned} T &= \frac{\sqrt{\pi}}{2^s l! (p-l)! (\frac{s}{2} - p)!} \sum_{m=0}^{\frac{s}{2}-p} (-1)^m \binom{\frac{s}{2}-p}{m} \\ &= \frac{\sqrt{\pi}}{2^s l! (p-l)! (\frac{s}{2} - p)!} (1-1)^{\frac{s}{2}-p} \\ &= \mathbf{1}\{p = \frac{s}{2}\} \frac{\sqrt{\pi}}{2^s l! (\frac{s}{2} - l)!}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} S &= \frac{(k-1)\sqrt{\pi}}{2^s l! (\frac{s}{2} - l)!} \sum_{q=(z-\frac{s}{2})^+}^0 (-1)^{l+q-z} \binom{\frac{s}{2} + q - l}{z-l} \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{k+1}{2} - q)}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2} - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+s-1}{2} + q - l)} \\ &= (-1)^{l-z} \frac{(k-1)\sqrt{\pi} \Gamma(\frac{1}{2})}{2^s l! (\frac{s}{2} - l)!} \binom{\frac{s}{2} - l}{z-l} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2} - l) \Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+s-1}{2} - l)} \\ &= (-1)^{l-z} \frac{\Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+1}{2})} \frac{(k-1)\pi}{2^s l! (\frac{s}{2} - l)!} \binom{\frac{s}{2} - l}{z-l} \frac{\Gamma(\frac{k+s-1}{2} - l)}{\Gamma(\frac{n+s-1}{2} - l)}. \end{aligned}$$

Furthermore, Legendre's duplication formula yields

$$s!S = (-1)^{l-z} \frac{(k-1)\sqrt{\pi} \Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2}) \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})} \underbrace{\binom{\frac{s}{2}}{l} \binom{\frac{s}{2} - l}{z-l}}_{= \binom{\frac{s}{2}}{z} \binom{z}{l}} \frac{\Gamma(\frac{k+s-1}{2} - l)}{\Gamma(\frac{n+s-1}{2} - l)}.$$

Thus, we obtain

$$\begin{aligned} &\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})} \frac{(k-1) \Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2}) \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})} \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \\ &\quad \times \sum_{l=0}^z (-1)^{l-z} \binom{z}{l} \frac{\Gamma(\frac{k+s-1}{2} - l)}{\Gamma(\frac{n+s-1}{2} - l)} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta). \end{aligned}$$

From Lemma 7 we conclude

$$\begin{aligned}
& \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
&= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})} \frac{(k-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+s-1}{2})} \\
&\quad \times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\frac{k+s-1}{2} - z) \Gamma(\frac{n-k}{2} + z) Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).
\end{aligned}$$

With

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}$$

we get

$$\begin{aligned}
& \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
&= \frac{(n-2)!}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!} \frac{\Gamma(\frac{k+1}{2})}{(k-2)!} \frac{\pi^{\frac{n-k-1}{2}} \Gamma(\frac{s+1}{2})}{2\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})} \\
&\quad \times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\frac{k+s-1}{2} - z) \Gamma(\frac{n-k}{2} + z) Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).
\end{aligned}$$

Legendre's formula applied three times gives

$$\begin{aligned}
& \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
&= \frac{k-1}{n-1} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})} \\
&\quad \times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\frac{k+s-1}{2} - z) \Gamma(\frac{n-k}{2} + z) Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),
\end{aligned}$$

which confirms the coefficients for even  $s$ .

On the other hand, if  $s$  is odd, then Lemma 8 yields

$$\begin{aligned}
T &= \frac{\sqrt{\pi}}{2^s l! (p-l)!} \sum_{m=0}^{\frac{s-1}{2}-p} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - p - m + 1)} \\
&= \frac{\sqrt{\pi}}{2^s l! (p-l)!} \left( \sum_{m=0}^{\frac{s+1}{2}-p} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - p - m + 1)} - (-1)^{\frac{s+1}{2}-p} \frac{1}{(\frac{s+1}{2} - p)! \Gamma(\frac{1}{2})} \right) \\
&= \frac{\sqrt{\pi}}{2^s l! (p-l)!} \left( (-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi} (-s+2p) (\frac{s+1}{2} - p)!} - (-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi} (\frac{s+1}{2} - p)!} \right) \\
&= (-1)^{\frac{s-1}{2}-p} \frac{\sqrt{\pi}}{2^s l! (p-l)!} \frac{1}{\sqrt{\pi} (\frac{s+1}{2} - p)!} \left( \frac{1}{s-2p} + 1 \right) \\
&= (-1)^{\frac{s-1}{2}-p} \frac{1}{2^{s-1} (s-2p) (\frac{s-1}{2} - p)! l! (p-l)!} \\
&= (-1)^{\frac{s-1}{2}-p} \frac{2\Gamma(\frac{s}{2} + 1)}{\sqrt{\pi} (s-2p) s!} \binom{\frac{s-1}{2}}{p} \binom{p}{l}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
s! \sum_{l=0}^z S &= \frac{2(k-1)\Gamma(\frac{s}{2} + 1)}{\sqrt{\pi}} \sum_{l=0}^z \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \\
&\quad \times \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q + \frac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{k+s+1}{2} - p - q) \Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+s+1}{2} - p) \Gamma(\frac{n-1}{2} + p + q - l)}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
&= 2(k-1) \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k-1}{2}} \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\frac{s-1}{2}} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta) \\
&\quad \times \sum_{l=0}^z \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \\
&\quad \times \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - p - q) \Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+s+1}{2} - p) \Gamma(\frac{n-1}{2} + p + q - l)}.
\end{aligned}$$

With

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}$$

we get

$$\begin{aligned} & \int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \frac{(n-2)!}{(n-k-1)!(k-2)!} \frac{\pi^{\frac{n-k-3}{2}} \Gamma(\frac{n-k+1}{2}) \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\frac{s-1}{2}} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta) \\ & \times \sum_{l=0}^z \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \\ & \times \Gamma(q + \frac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - p - q) \Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+s+1}{2} - p) \Gamma(\frac{n-1}{2} + p + q - l)}. \end{aligned}$$

We denote the threefold sum with respect to  $l$ ,  $p$  and  $q$  by  $R$ . Hence,  $R$  multiplied with the factor in front of the sum with respect to  $z$  equals  $\kappa_{n,k,s,z}$ . A direct calculation for  $R$  still remains an open task. However, for the proof this is not required.

Finally, if  $s$  is odd we calculate the only so far unknown coefficient  $\kappa_{n,k,s,(s-1)/2}$ . For  $z = (s-1)/2$  we see that the sum over  $q$  only contains one summand, namely  $q = (s-1)/2 - p$ . Hence, we obtain

$$\begin{aligned} R &= \Gamma(\frac{k}{2} + 1) \sum_{l=0}^z \sum_{p=l}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+l} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \Gamma(\frac{s}{2} - p) \frac{\Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k+s-1}{2} - p)}{\Gamma(\frac{n+s+1}{2} - p) \Gamma(\frac{n+s}{2} - l - 1)} \\ &= \Gamma(\frac{k}{2} + 1) \sum_{p=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} \binom{\frac{s-1}{2}}{p} \Gamma(\frac{s}{2} - p) \frac{\Gamma(\frac{n-k+s-1}{2} - p)}{\Gamma(\frac{n+s+1}{2} - p)} \sum_{l=0}^p (-1)^l \binom{p}{l} \frac{\Gamma(\frac{k-1}{2} + p - l)}{\Gamma(\frac{n+s}{2} - l - 1)}. \end{aligned}$$

Then Lemma 7 yields

$$\begin{aligned} R &= \frac{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{k-1}{2}) \Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2} - 1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+p} \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{s}{2} - p)}{\Gamma(\frac{n+s+1}{2} - p)} \\ &= \frac{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{k-1}{2}) \Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2} - 1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^p \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(\frac{n}{2} + 1 + p)}. \end{aligned}$$

Again, we apply Lemma 7 and obtain

$$R = \sqrt{\pi} \frac{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+s}{2}) \Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2} - 1) \Gamma(\frac{n+s+1}{2})}.$$

Thus, we conclude

$$\begin{aligned}\kappa_{n,k,s,\frac{s-1}{2}} &= \frac{(n-2)!}{(n-k-1)!(k-2)!} \frac{\pi^{\frac{n-k-3}{2}} \Gamma(\frac{n-k+1}{2}) \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2})} R \\ &= \pi^{\frac{n-k-2}{2}} \frac{(n-2)!}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{k-1}{2})}{(k-2)!} \frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!} \frac{(n+s-2) \Gamma(\frac{s}{2} + 1)}{(n-k+s-1) \Gamma(\frac{n+s+1}{2})}.\end{aligned}$$

Applying three times Legendre's formula gives

$$\kappa_{n,k,s,\frac{s-1}{2}} = \pi^{\frac{n-k-1}{2}} \frac{2k(n+s-2)}{(n-1)(n-k+s-1)} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n+s+1}{2})},$$

which completes the argument.  $\square$

Next we prove Theorem 6. As in the previous proof, one can compare the Crofton integral to the global one obtained in [3, Theorem 3]. However, we deduce it directly from Theorem 3.

*Proof (Theorem 6).* Lemma 6 yields

$$\begin{aligned}& \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ &= \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} Q^l \\ & \quad \times \int_{A(n,1)} Q(E)^{m-l} \phi_{0,E}^{r,s-2m,0}(K \cap E, \beta \cap E) \mu_1(dE).\end{aligned}$$

If  $s \in \mathbb{N}_0$  is even, we conclude from Theorem 3

$$\begin{aligned}& \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ &= \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\frac{s}{2}} \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \\ & \quad \times \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s+1}{2} - l)}{\pi \Gamma(\frac{n+s+1}{2} - l)} \sum_{z=0}^{\frac{s}{2}-l} (-1)^z \binom{\frac{s}{2}-l}{z} \frac{1}{1-2z} Q^{\frac{s}{2}-z} \phi_{n-1}^{r,2z,0}(K, \beta).\end{aligned}$$

A change of the order of summation yields

$$\begin{aligned}
& \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\
&= \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s}{2}} \sum_{m=l}^{\frac{s}{2}} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \\
&\quad \times \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s+1}{2} - l)}{\pi \Gamma(\frac{n+s+1}{2} - l)} \sum_{z=0}^{\frac{s}{2}-l} (-1)^z \binom{\frac{s}{2}-l}{z} \frac{1}{1-2z} Q^{\frac{s}{2}-z} \phi_{n-1}^{r,2z,0}(K, \beta).
\end{aligned}$$

Legendre's duplication formula gives for the sum with respect to  $m$ , which we denote by  $S$ ,

$$\begin{aligned}
S &= \frac{\sqrt{\pi}}{2^s} \sum_{m=l}^{\frac{s}{2}} (-1)^{m-l} \binom{m}{l} \frac{1}{m! \Gamma(\frac{s}{2} - m + 1)} \\
&= \frac{\sqrt{\pi}}{2^s l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - l - m + 1)}.
\end{aligned}$$

As seen before, we conclude from the binomial theorem

$$\begin{aligned}
S &= \frac{\sqrt{\pi}}{2^s (\frac{s}{2} - l)! l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \binom{\frac{s}{2}-l}{m} \\
&= \mathbf{1}\{l = \frac{s}{2}\} \frac{\Gamma(\frac{s+1}{2})}{s!}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) &= \frac{\pi^{\frac{n-3}{2}} \Gamma(\frac{s+1}{2}) \Gamma(\frac{n}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{n+s}{2}) \Gamma(\frac{n+1}{2})} Q^{\frac{s}{2}} \phi_{n-1}^{r,0,0}(K, \beta) \\
&= \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s}{2}) \Gamma(\frac{n+1}{2})} Q^{\frac{s}{2}} \phi_{n-1}^{r,0,0}(K, \beta).
\end{aligned}$$

On the other hand, if  $s \in \mathbb{N}$  is odd, we conclude from Theorem 3

$$\begin{aligned}
& \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\
&= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\frac{s-1}{2}} \sum_{l=0}^m (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} \frac{\Gamma(\frac{s}{2} - l + 1)}{\Gamma(\frac{n+s+1}{2} - l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K, \beta).
\end{aligned}$$

A change of the order of summation yields

$$\begin{aligned} & \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ &= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} \sum_{m=l}^{\frac{s-1}{2}} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m m! (s-2m)!} \binom{m}{l} \frac{\Gamma(\frac{s}{2} - l + 1)}{\Gamma(\frac{n+s+1}{2} - l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K, \beta). \end{aligned}$$

Legendre's duplication formula gives for the sum with respect to  $m$ , which we denote by  $S$ ,

$$S = \frac{\sqrt{\pi}}{2^s l!} \sum_{m=0}^{\frac{s-1}{2}-l} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - l - m + 1)}.$$

Then Lemma 8 yields

$$\begin{aligned} S &= \frac{\sqrt{\pi}}{2^s l!} \left( \sum_{m=0}^{\frac{s+1}{2}-l} (-1)^m \frac{1}{m! \Gamma(\frac{s}{2} - l - m + 1)} - (-1)^{\frac{s+1}{2}-l} \frac{1}{(\frac{s+1}{2} - l)! \Gamma(\frac{1}{2})} \right) \\ &= \frac{\sqrt{\pi}}{2^s l!} \left( (-1)^{\frac{s-1}{2}-l} \frac{1}{\sqrt{\pi} (s-2l) (\frac{s+1}{2} - l)!} - (-1)^{\frac{s+1}{2}-l} \frac{1}{\sqrt{\pi} (\frac{s+1}{2} - l)!} \right) \\ &= (-1)^{\frac{s-1}{2}-l} \frac{1}{2^{s-1} l! (s-2l) (\frac{s-1}{2} - l)!}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ &= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{2^s \Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}-l} \frac{1}{l! (\frac{s-1}{2} - l)!} \frac{\Gamma(\frac{s}{2} - l)}{\Gamma(\frac{n+s+1}{2} - l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K, \beta) \\ &= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{2^s \Gamma(\frac{s+1}{2}) \Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^l \binom{\frac{s-1}{2}}{l} \frac{\Gamma(l + \frac{1}{2})}{\Gamma(\frac{n+2}{2} + l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K, \beta). \end{aligned}$$

Then Lemma 7 gives

$$\begin{aligned} & \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_1(dE) \\ &= \frac{s!}{2^s \Gamma(\frac{s+1}{2})} \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{n+s+1}{2}) \Gamma(\frac{n+1}{2})} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K, \beta). \end{aligned}$$

Finally, the assertion follows from Legendre's duplication formula.  $\square$

Finally, we show that the Crofton formula has a very simple form in the  $\psi$ -representation of tensorial curvature measures.



*Proof of Corollary 7.* The cases  $s \in \{0, 1\}$  are checked directly, hence we can assume  $s \geq 2$  in the following. Using (9) we get

$$\begin{aligned}
 & \int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
 &= \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + s - 1)} Q^j \\
 (17) \quad & \times \int_{A(n,k)} \phi_{k-1}^{r,s-2j,0}(K \cap E, \beta \cap E) \mu_k(dE).
 \end{aligned}$$

Then, for  $k \neq 1$ , Theorem 5 yields

$$\begin{aligned}
 & \int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
 &= \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + s - 1)} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - j} \kappa_{n,k,s-2j,z} Q^{z+j} \phi_{n-1}^{r,s-2j-2z,0}(K, \beta) \\
 &= \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{z=j}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + s - 1)} \kappa_{n,k,s-2j,z-j} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),
 \end{aligned}$$

where

$$\kappa_{n,k,s-2j,z-j} = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2} - j) \Gamma(\frac{s}{2} - j + 1)}{\Gamma(\frac{n-k+s+1}{2} - j) \Gamma(\frac{n+s-1}{2} - j)} \frac{\Gamma(\frac{n-k}{2} + z - j) \Gamma(\frac{k+s-1}{2} - z)}{\Gamma(\frac{s}{2} - z + 1) (z-j)!},$$

if  $z \neq (s-1)/2$ . On the other hand, if  $z = (s-1)/2$ , then the coefficient needs to be multiplied by the factor  $\frac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)}$  (see the comment after the proof of Theorem 5).

Applying Legendre's duplication formula twice, we thus obtain

$$\begin{aligned}
 & \int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
 &= \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} \frac{s!}{2^s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma(\frac{k+s-1}{2} - z)}{z! \Gamma(\frac{n}{2} + s - 1) \Gamma(\frac{s}{2} - z + 1)} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta) \\
 & \quad \times \sum_{j=0}^z (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2} + s - j - 1) \Gamma(\frac{n-k}{2} + z - j)}{\Gamma(\frac{n-k+s+1}{2} - j) \Gamma(\frac{n+s-1}{2} - j)} \\
 & \quad \times \left( 1 - \mathbf{1}\{z = \frac{s-1}{2}\} \left( 1 - \frac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)} \right) \right),
 \end{aligned}$$

Denoting the sum with respect to  $j$  by  $S_z$ , an application of Lemma 9 shows that

$$\begin{aligned}
 S_z &= \sum_{j=0}^z (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2} + s - j - 1) \Gamma(\frac{n-k}{2} + z - j)}{\Gamma(\frac{n-k+s+1}{2} - j) \Gamma(\frac{n+s-1}{2} - j)} \\
 (18) \quad &= (-1)^z \frac{\Gamma(\frac{n-k}{2}) \Gamma(\frac{s+1}{2}) \Gamma(\frac{k+s-1}{2}) \Gamma(\frac{n}{2} + s - z - 1)}{\Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2}) \Gamma(\frac{s+1}{2} - z) \Gamma(\frac{k+s-1}{2} - z)},
 \end{aligned}$$

for  $z \neq (s-1)/2$  and  $k > 1$ . On the other hand, for  $z = (s-1)/2 =: t$ , we obtain from Lemma 9 and Lemma 10 (since  $s > 1$  and thus  $t > 0$ ) that

$$\begin{aligned}
 S_t &= \frac{k}{k-1} \sum_{j=0}^t (-1)^j \binom{t}{j} \left(1 - \frac{1}{n+2t-2j}\right) \frac{\Gamma(\frac{n}{2} + 2t - j) \Gamma(\frac{n-k}{2} + t - j)}{\Gamma(\frac{n-k}{2} + t - j + 1) \Gamma(\frac{n}{2} + t - j)} \\
 &= \frac{k}{k-1} \left( \sum_{j=0}^t (-1)^j \binom{t}{j} \frac{\Gamma(\frac{n}{2} + 2t - j) \Gamma(\frac{n-k}{2} + t - j)}{\Gamma(\frac{n-k}{2} + t - j + 1) \Gamma(\frac{n}{2} + t - j)} \right. \\
 &\quad \left. - \sum_{j=0}^t (-1)^j \binom{t}{j} \frac{1}{\frac{n-k}{2} + t - j} \frac{\Gamma(\frac{n}{2} + 2t - j)}{\Gamma(\frac{n}{2} + t - j + 1)} \right) \\
 &= (-1)^t \frac{\Gamma(\frac{n-k}{2}) \Gamma(t+1) \Gamma(\frac{k}{2} + t)}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2} + t + 1)},
 \end{aligned}$$

which coincides with (18) for  $z = (s-1)/2$ .

Thus, we have

$$\begin{aligned}
 &\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
 &= \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2})} \frac{s! \Gamma(\frac{s+1}{2})}{2^s} \\
 &\quad \times \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \frac{\Gamma(\frac{n}{2} + s - z - 1)}{z! \Gamma(\frac{n}{2} + s - 1) \Gamma(\frac{s}{2} - z + 1) \Gamma(\frac{s+1}{2} - z)} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).
 \end{aligned}$$

Applying Legendre's duplication formula twice, we get

$$\begin{aligned}
 &\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\
 &= \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{k+s-1}{2}) \Gamma(\frac{s+1}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2})} \\
 &\quad \times \frac{1}{\sqrt{\pi}} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \binom{s}{2z} \frac{\Gamma(z + \frac{1}{2}) \Gamma(\frac{n}{2} + s - z - 1)}{\Gamma(\frac{n}{2} + s - 1)} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).
 \end{aligned}$$

With (9) we obtain the assertion for  $k \neq 1$ .

On the other hand, if  $k = 1$ , then Theorem 6 yields for (17) that

$$\begin{aligned} & \int_{A(n,1)} \psi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \frac{\pi^{\frac{n-3}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2}) \Gamma(\frac{n}{2} + s - 1)} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j + \frac{1}{2}) \Gamma(\frac{n}{2} + s - j - 1) \Gamma(\lfloor \frac{s+1}{2} \rfloor - j + \frac{1}{2})}{\Gamma(\frac{n}{2} + \lfloor \frac{s+1}{2} \rfloor - j)} \\ & \quad \times Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta). \end{aligned}$$

Denoting the sum with respect to  $j$  by  $S$  and applying Legendre's duplication formula three times, we conclude that

$$S = \sqrt{\pi} \Gamma(\lfloor \frac{s+1}{2} \rfloor + \frac{1}{2}) \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{\lfloor \frac{s}{2} \rfloor}{j} \frac{\Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + \lfloor \frac{s+1}{2} \rfloor - j)}.$$

Since  $s \geq 2$ , Lemma 7 yields  $S = 0$  due to (19), and hence the assertion.  $\square$

## 6. SUMS OF GAMMA FUNCTIONS

In this section, we state four basic identities involving sums of Gamma functions.

**Lemma 7.** *Let  $q \in \mathbb{N}_0$  and  $a, b > 0$ . Then*

$$\sum_{y=0}^q (-1)^y \binom{q}{y} \frac{\Gamma(a+y)}{\Gamma(b+y)} = \frac{\Gamma(a) \Gamma(b-a+q)}{\Gamma(b+q) \Gamma(b-a)}.$$

Under the additional assumption  $a < b$ , this lemma can be found as Lemma 15.6.4 in [1], which is also proved there. Since this case is not sufficient for our purposes, we deduce the current more general version via Zeilberger's algorithm.

The factor  $\Gamma(b-a+q)$  in Lemma 7 does not cause any problems in case  $a-b-q \in \mathbb{N}_0$ , as the also appearing  $\Gamma(b-a)$  cancels out the singularity, see (19).

*Proof.* We set

$$F(q, y) := (-1)^y \binom{q}{y} \frac{\Gamma(a+y)}{\Gamma(b+y)},$$

for which we see that  $F(q, y) = 0$  if  $y \notin \{0, \dots, q\}$ , and

$$f(q) := \sum_{y=0}^q F(q, y).$$

Furthermore, we define the function

$$G(q, y) := \begin{cases} \frac{y(b+y-1)}{q-y+1} F(q, y), & \text{for } y \in \{0, \dots, q\}, \\ G(q, q) - (b+q)F(q+1, q) \\ \quad + (b-a+q)F(q, q), & \text{for } y = q+1, \\ 0, & \text{else.} \end{cases}$$

A direct calculation yields

$$-(b+q-1)F(q, y) + (b-a+q-1)F(q-1, y) = G(q-1, y+1) - G(q-1, y)$$

for  $y \in \mathbb{N}_0$ . Summing this relation over  $y \in \{0, \dots, q\}$  gives

$$-(b+q-1)f(q) + (b-a+q-1)f(q-1) = 0$$

and thus

$$\begin{aligned} f(q) &= \frac{b-a+q-1}{b+q-1} f(q-1) \\ &= \frac{(b-a+q-2)(b-a+q-1)}{(b+q-2)(b+q-1)} f(q-2) \\ &\vdots \\ &= \frac{(b-a) \cdots (b-a+q-1)}{b \cdots (b+q-1)} f(0) \\ &= \frac{\Gamma(b-a+q)\Gamma(b)}{\Gamma(b+q)\Gamma(b-a)} f(0), \end{aligned}$$

where

$$(19) \quad \frac{\Gamma(b-a+q)}{\Gamma(b-a)} = (b-a) \cdots (b-a+q-1)$$

is well-defined, even for  $a-b \in \mathbb{N}$ . With

$$f(0) = \frac{\Gamma(a)}{\Gamma(b)}$$

we obtain the assertion. □

**Lemma 8.** *Let  $a \in \mathbb{N}_0$ . Then*

$$\sum_{q=0}^a \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} = \frac{(-1)^a}{\sqrt{\pi}(1-2a)a!}.$$

*Proof.* For the sum  $S$  on the left-hand side of the asserted equation, we obtain

$$S = \sum_{q=0}^a \left( \frac{2q}{2a-1} \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} + \frac{2q+2}{2a-1} \frac{(-1)^q}{\Gamma(a-q-\frac{1}{2})(q+1)!} \right),$$

where we use that  $(-\frac{1}{2})\Gamma(-\frac{1}{2}) = \sqrt{\pi}$ . Due to cancellation in this telescoping sum, the assertion follows immediately.  $\square$

Finally, we establish the following lemmas.

**Lemma 9.** *Let  $a, b, c \in \mathbb{R}$  and  $z \in \mathbb{N}_0$  with  $a > z \geq 0$  and  $b > 0$ . Then*

$$\begin{aligned} & \sum_{j=0}^z (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)} \\ &= (-1)^z \frac{\Gamma(a-z)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(c)} \frac{\Gamma(a-c+1)}{\Gamma(a-c+1-z)} \frac{\Gamma(c-b)}{\Gamma(c-b-z)}. \end{aligned}$$

The factor  $\Gamma(a-c+1)$  (resp.  $\Gamma(c-b)$ ) in Lemma 9 does not cause any problems for  $c-a \in \mathbb{N}$  (resp.  $b-c \in \mathbb{N}_0$ ), as the also appearing  $\Gamma(a-c+1-z)$  (resp.  $\Gamma(c-b-z)$ ) cancels out the singularity. On the other hand, in our applications of the lemma, we only need the cases where  $a-c+1 > z$  and  $c-b > z$ .

*Proof.* We set

$$F(z, j) := (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)},$$

for  $j \in \{0, \dots, z\}$ , and  $F(z, j) = 0$  in all other cases, and

$$f(z) := \sum_{j=0}^z F(z, j).$$

Furthermore, we define the function

$$G(z, j) := \begin{cases} -\frac{j(a-j)(b+z-j)}{z-j+1} F(z, j), & \text{for } j \in \{0, \dots, z\}, \\ G(z, z) + (a-z-1)F(z+1, z) \\ \quad + (c-b-z-1)(a-c-z)F(z, z), & \text{for } j = z+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$(a-z)F(z, j) + (c-b-z)(a-c-z+1)F(z-1, j) = G(z-1, j+1) - G(z-1, j)$$

for  $j \in \mathbb{N}_0$ . Summing this relation over  $j \in \{0, \dots, z\}$  gives

$$(a-z)f(z) + (c-b-z)(a-c-z+1)f(z-1) = 0$$

and thus

$$\begin{aligned}
f(z) &= -\frac{(c-b-z)(a-c-z+1)}{a-z}f(z-1) \\
&= \frac{(c-b-z)(c-b-z+1)(a-c-z+1)(a-c-z+2)}{(a-z)(a-z+1)}f(z-2) \\
&\vdots \\
&= (-1)^z \frac{(c-b-z) \cdots (c-b-1)(a-c-z+1) \cdots (a-c)}{(a-z) \cdots (a-1)}f(0) \\
&= (-1)^z \frac{\Gamma(c-b)\Gamma(a-c+1)\Gamma(a-z)}{\Gamma(c-b-z)\Gamma(a-c+1-z)\Gamma(a)}f(0),
\end{aligned}$$

where

$$\frac{\Gamma(c-b)}{\Gamma(c-b-z)} = (c-b-z) \cdots (c-b-1)$$

is well-defined, even for  $b-c \in \mathbb{N}_0$ , and a similar statement holds for  $\Gamma(a-c+1)/\Gamma(a-c+1-z)$ . With

$$f(0) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c+1)}$$

we obtain the assertion. □

**Lemma 10.** *Let  $a, b \in \mathbb{R}$  with  $a, b > 0$  and  $t \in \mathbb{N}$ . Then*

$$\sum_{j=0}^t (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)} = \frac{\Gamma(a-b+t)\Gamma(b)\Gamma(t+1)}{\Gamma(a-b+1)\Gamma(b+t+1)}.$$

The factor  $\Gamma(a-b+t)$  in Lemma 10 does not cause any problems for  $b-a-t \in \mathbb{N}_0$ , as the also appearing  $\Gamma(a-b+1)$  cancels out the singularity. In our application of the lemma, we will additionally know that  $a > b$ .

*Proof.* We set

$$F(t, j) := (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)},$$

for which we see that  $F(t, j) = 0$  if  $j \notin \{0, \dots, t\}$ , and

$$f(t) := \sum_{j=0}^t F(t, j).$$

Furthermore, we define the function

$$G(t, j) := \begin{cases} \frac{j(a+j)(a+2t+1)(t^2+t(a+1)-j+1)(b+j)}{t(t-j+1)(a+t)(a+t+1)} F(t, j), & \text{for } j \in \{0, \dots, t\}, \\ G(t, t) - (b+t+1)F(t+1, t) \\ \quad + (t+1)(a-b+t)F(t, t), & \text{for } j = t+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$-(b+t)F(t, j) + t(a-b+t-1)F(t-1, j) = G(t-1, j+1) - G(t-1, j)$$

for  $j \in \mathbb{N}_0$ . Summing this relation over  $j \in \{0, \dots, t\}$  gives

$$-(b+t)f(t) + t(a-b+t-1)f(t-1) = 0$$

and thus

$$\begin{aligned} f(t) &= \frac{t(a-b+t-1)}{b+t} f(t-1) \\ &= \frac{(t-1)t(a-b+t-2)(a-b+t-1)}{(b+t-1)(b+t)} f(t-2) \\ &\vdots \\ &= \frac{2 \cdots t(a-b+1) \cdots (a-b+t-1)}{(b+2) \cdots (b+t)} f(1) \\ &= \frac{\Gamma(t+1)\Gamma(a-b+t)\Gamma(b+2)}{\Gamma(a-b+1)\Gamma(b+t+1)} f(1). \end{aligned}$$

With

$$f(1) = \frac{1}{b} - \frac{1}{b+1} = \frac{1}{b(b+1)}$$

we obtain the assertion.  $\square$

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KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT), DEPARTMENT OF MATHEMATICS, D-76128 KARLSRUHE, GERMANY

*E-mail address:* daniel.hug@kit.edu

KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT), DEPARTMENT OF MATHEMATICS, D-76128 KARLSRUHE, GERMANY

*E-mail address:* jan.weis@kit.edu